Number of times a site is visited in two-dimensional random walks

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In this paper, formulas are derived to compute the mean number of times a site has been visited in a random walk on a two-dimensional lattice. Asymmetric random walks are considered, with or without drift, for different boundary conditions. It is shown that in case of absorbing boundaries the mean number of visits reaches stationary values over the lattice; comparisons with a Monte Carlo simulation are also presented.

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I. INTRODUCTION

The theory of random walks on lattices, besides being a central part of Markov chain theory, has applications in a variety of research fields, such as theory of potential [1], statistical field theory [2], biophysics [3], and population biology [4,5], just to name a few.

An interesting class of problems in the random walk theory arises by considering the statistics of the visits of a random walker to the sites. For instance, one can look at the number of distinct sites visited by a n-step walk, [6]; this problem has been studied in detail by Larralde *et al.* [7–9].

A different, but complementary, approach is taken here, and it consists of looking at the number of times a site is visited by a random walker. Some general definitions and properties of the mean number of times a site is visited can be found in Ref. [1], whereas in case of simple (symmetric) random walk, asymptotic results have been provided by Ref. [6]; furthermore, results from a Monte Carlo simulation have been reported in Ref. [10] in the case of a two-dimensional simple (symmetric) random walk.

In this paper exact formulas will be derived for the mean number of times a site has been visited if the random walk is not symmetric, i.e., the transition probability depends on the direction of the step. In particular, random walks with drift and random walks with different diffusion coefficients along the coordinate axes will be considered.

The number of visits to a site is closely related to potential theory [1]; furthermore, it is interesting to note that it can be interpreted as a trace, or a memory, left by the random walk on the lattice. On the other hand, asymmetric random walks are relevant in many natural phenomena as, for instance, in animal taxis and dispersal in biology [5,4] or motion of particles in sedimentation or electrophoresis [11].

II. PRELIMINARIES

The random walk takes place on a lattice S. Let $\mathbf{s}=(x,y)$ be a site of S with coordinates (x,y) and let $P_k(\mathbf{s})$ be the probability that after k steps the walker, starting from the origin, is at the site $\mathbf{s}$. In what follows, the notation is the same as in Ref. [6]; denote by $\beta_n^{(r)}(\mathbf{s})$ the probability that, at time n, the site $\mathbf{s}$ has been visited exactly r times.

It is immediately seen that $\beta_n^{(r)}(\mathbf{s})$ must satisfy the relation

$$\beta_n^{(r)}(\mathbf{s})=\beta_{n-1}^{(r-1)}(\mathbf{s})P_n(\mathbf{s})+\beta_{n-1}^{(r)}(\mathbf{s})[1-P_n(\mathbf{s})], \tag{2.1}$$

thus $\beta_n^{(r)}(\mathbf{s})$ can be computed iteratively once the initial conditions have been fixed, which for a walker starting from $\mathbf{s}=(0,0)$, are $\beta_0^0=(1-\delta_{\mathbf{s}0})$ and $\beta_0^1=\delta_{\mathbf{s}0}$.

The mean number of visits is

$$M_n(\mathbf{s})=\sum_r r\beta_n^{(r)}(\mathbf{s}), \tag{2.2}$$

and hence, from Eq. (2.1)

$$M_n(\mathbf{s})=P_n(\mathbf{s})\sum_{r=0}^{\infty} r\beta_{n-1}^{(r-1)}(\mathbf{s})+[1-P_n(\mathbf{s})]\sum_{r=0}^{\infty} r\beta_{n-1}^{(r)}(\mathbf{s}). \tag{2.3}$$

Note that $\Sigma_{r=0}^{\infty} r\beta_{n-1}^{(r)}(\mathbf{s})=M_{n-1}(\mathbf{s})$; by setting $r=l+1$, and making use again of Eq. (2.1) one obtains

$$M_n(\mathbf{s})=[1-P_n(\mathbf{s})]M_{n-1}(\mathbf{s})+P_n(\mathbf{s})\sum_{l=0}^{\infty} l\beta_{n-1}^{(l)}(\mathbf{s})+P_n(\mathbf{s})\sum_{l=0}^{\infty}\beta_{n-1}^{(l)}(\mathbf{s}). \tag{2.4}$$

Since $\Sigma_{l=0}^{\infty} l\beta_{n-1}^{(l)}(\mathbf{s})=M_{n-1}(\mathbf{s})$ and, furthermore, $\Sigma_{l=0}^{\infty}\beta_{n-1}^{(l)}(\mathbf{s})=1$, Eq. (2.4) becomes

$$M_n(\mathbf{s})=M_{n-1}(\mathbf{s})+P_n(\mathbf{s}). \tag{2.5}$$

Equation (2.5) can be easily solved and the result is

$$M_n(\mathbf{s})=\sum_{k=0}^{n} P_k(\mathbf{s}), \tag{2.6}$$

where it is assumed $P_0(\mathbf{s})=M_0(\mathbf{s})=\delta_{\mathbf{s},0}$.

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Finally, we define

$$\mathcal{M}=\lim_{n\to\infty} M_n . \tag{2.7}$$

Equation (2.6) does not depend on any specific form of the probability P_k, and can then be applied to different types of random walks.

In the sequel it will be supposed that in a single step the walker can move from a site $\mathbf{s}=(x,y)$ to any of the sites with coordinates $(x,y+1),(x,y-1)(x+1,y),(x-1,y)$. Single steps of the walker will be denoted by δx or δy, as appropriate: thus $\delta x,\delta y=\pm 1$. The transition probabilities are assumed to be independent from time and denoted by $p(x-1,y|x,y)$, $p(x+1,y|x,y)$, $p(x,y-1|x,y)$, $p(x,y+1|x,y)$. For simplicity, sometimes the compact notations $p_l=p(x-1,y|x,y)$, $p_r=p(x+1,y|x,y)$, $q_d=p(x,y-1|x,y)$, $q_u=p(x,y+1|x,y)$ will also be used.

The probability $P_k(\mathbf{s})$ can be computed by a straightforward generalization of the one-dimensional case. At time k let r_1, r_2 be the steps taken by the walker, in the positive and negative direction, respectively, of the x axis; similarly let l_1 and l_2 be the steps taken along the y axis. For the walker to be at the site $\mathbf{s}=(x,y)$ the following conditions must be satisfied:

$$x=r_1-r_2,\quad y=l_1-l_2,\quad k=r_1+r_2+l_1+l_2. \tag{2.8}$$

Then the formula for $P_k(\mathbf{s})$ in the case of unrestricted random walks is

$$P_k(\mathbf{s})=\sum_{r_1,r_2,l_1,l_2}\frac{k!}{r_1!r_2!l_1!l_2!}p_r^{r_1}p_l^{r_2}q_u^{l_1}q_d^{l_2}, \tag{2.9}$$

where the sum is taken on all r_i, l_i satisfying relations (2.8). Obviously $P_k(\mathbf{s})=0$ if $|x|+|y|>k$ or if k and $|x|+|y|$ do not have the same parity [12].

If the random walk is restricted, i.e., takes place on a bounded domain, the appropriate boundary conditions must also be taken into account. Let ∂S be the boundary of S: if ∂S is absorbing, the random walk stops once the walker has reached the boundary, whereas if it is reflecting the walker is forced to return to the last position occupied before hitting the boundary [13].

In any case Eq. (2.9) is not very useful in practice, since P_k is very difficult to calculate even for moderately large values of k. Fortunately, there exist well-known approximations to Eq. (2.9): for instance in the case of unrestricted random walks the Gaussian, or normal, approximation is widely used [1,13], whereas, by using, in place of k, the continuous variable t and by assuming continuity of x, y, the diffusion approximation [14] allows to replace $P_k(\mathbf{s})$ with the solution $P(x,y;t)$ of the Fokker-Planck equation [13] that, in the present case, is

$$\frac{\partial P}{\partial t}=\frac{1}{2}\sigma_x^2\frac{\partial^2 P}{\partial x^2}+\frac{1}{2}\sigma_y^2\frac{\partial^2 P}{\partial y^2}-u_x\frac{\partial P}{\partial x}-u_y\frac{\partial P}{\partial y}, \tag{2.10}$$

where $u_x=\langle \delta x\rangle$, $u_y=\langle \delta y\rangle$ are the averages of δx and δy, respectively, and $\sigma_x^2=\sigma^2(\delta x)$, $\sigma_y^2=\sigma^2(\delta y)$ the corresponding variances.

The mean number of visits, computed with the Gaussian approximation, will be still denoted by M_n, whereas in the case of the diffusion approximation $M_n(\mathbf{s})$ is replaced by $M(x,y;t)$ defined as

$$M(x,y;t)=\int_0^t P(x,y;\tau)d\tau, \tag{2.11}$$

and it is clear that $\mathcal{M}$ exists if and only if $\lim_{\tau\to\infty}P(\cdot;\tau)\tau=0$. It must be noted that Eqs. (2.6) and (2.11) operate at different time scales and that k can be considered a continuous variable only when large enough; then, $P(x,y,t)$ is a good approximation of $P_k(\mathbf{s})$ only when k is large. At points close to the origin $P(\cdot;\tau)$ and P_k contribute to $M(\cdot;t)$ and M_n, respectively, also for k and τ small, and, therefore, the agreement between Eqs. (2.6) and (2.11) should be poor. On the contrary, a closer agreement should be expected at points far from the origin, where P_k and $P(\cdot;\tau)$ contribute sensibly to M_k and $M(\cdot;t)$, respectively, only for k and τ large.

III. UNRESTRICTED RANDOM WALK

Consider an unrestricted random walk without drift, i.e., with $u_x=u_y=0$; Eq. (2.9) can be approximated by

$$P_k(\mathbf{s})=\frac{1}{2\pi k\sigma_x\sigma_y}\exp-\left\{\frac{x^2}{2k\sigma_x^2}+\frac{y^2}{2k\sigma_y^2}\right\}. \tag{3.1}$$

It is well known that the Gaussian approximation agrees with Eq. (2.9) only for large k (see, for instance, [1,13]), hence one must expect that M_n computed via Eq. (3.1) becomes an accurate approximation of the mean number of visits only when n is large.

The case of $M_n(0)$ will be first dealt with. Preliminarily note that, since $P_k(0)=0$ if k is odd, $M_{2n}=M_{2n+1}$, and, therefore, the mean number of visits at the origin is

$$M_{2n}(0)=\sum_{k=0}^{n}P_{2k}(0); \tag{3.2}$$

by using the Gaussian approximation

$$M_{2n}(0)=1+\frac{1}{2\pi\sigma_x\sigma_y}\sum_{k=1}^{n}\frac{1}{2k},\quad M_{2n}(0)=M_{2n+1}(0). \tag{3.3}$$

For n large

$$\sum_{k=1}^{n}\frac{1}{k}\sim\ln n, \tag{3.4}$$

[15]; then

$$M_n(0)\sim\frac{1}{4\pi\sigma_x\sigma_y}\ln n/2\sim\frac{1}{4\pi\sigma_x\sigma_y}\ln n. \tag{3.5}$$

Consider now $\mathbf{s}\neq 0$ and suppose $|x|+|y|$ to be even. Define $r^2=x^2/2\sigma_x^2+y^2/2\sigma_y^2$; M_{2n} is given by the formula

$$M_{2n}(\mathbf{s})=\frac{1}{2\pi\sigma_x\sigma_y}\sum_{k=1}^{n}\frac{1}{2k}\exp-(r^2/2k), \tag{3.6}$$

that can be written as

$$M_{2n}(\mathbf{s})=\frac{1}{4\pi\sigma_x\sigma_y}\left[\sum_{k=1}^{n}\frac{1}{k}-F_n(r^2)\right], \tag{3.7}$$

where

$$F_n(r^2)=\sum_{k=1}^{n}\frac{1}{k}[1-\exp-(r^2/2k)]. \tag{3.8}$$

M_{2n} decreases as r^2 increases, as expected, and the function $F_n(r^2)$ takes into account the delay with which the walker reaches sites different from the origin.

Moreover,

$$F(r^2)=\lim_{n\to\infty}F_n(r^2)=\sum_{k=1}^{\infty}\frac{1}{k}[1-\exp-(r^2/2k)], \tag{3.9}$$

exists and is finite for every finite r^2, as can be verified by performing a power series expansion of $F(r^2)$ and by noting that $\Sigma_{k=1}^{n}k^{-(j+1)}=\zeta(j+1)$, where ζ is the Riemann zeta function [16].

Thus, also at points $\mathbf{s}\neq 0$ the mean number of visits scales logarithmically as $n\to\infty$ [see Eqs. (3.4) and (3.7)]. The boundless growth in the case of unrestricted random walks with no drift should be expected because the simple random walk in two dimensions, i.e., the present case when $\sigma_x^2=\sigma_y^2$, is recurrent [17].

IV. RESTRICTED RANDOM WALK: ABSORBING BOUNDARIES

Let the random walk take place on a finite domain $S=[-a,a]\times[-a,a]\subset R^2$. In this case, the most straightforward way to compute the mean number of visits is via the solution of Eq. (2.10), since the analytic form of the probability P depends directly on the characteristics of the boundary.

If the boundary ∂S is absorbing, the solution of Eq. (2.10) is

$$\begin{aligned}P(x,y;t)=&\left(\frac{2}{L}\right)^2\exp\left[\left(\frac{u_x x}{\sigma_x^2}+\frac{u_y y}{\sigma_y^2}\right)-\left(\frac{u_x^2 t}{\sigma_x^2}+\frac{u_y^2 t}{\sigma_y^2}\right)\right]\\&\times\sum_{m,l=1}^{\infty}\exp-[(m^2\sigma_x^2+l^2\sigma_y^2)\pi^2 t/2L^2]\\&\times\sin\left[\frac{m\pi(x+a)}{L}\right]\sin\left[\frac{m\pi a}{L}\right]\sin\left[\frac{l\pi(y+a)}{L}\right]\\&\times\sin\left[\frac{l\pi a}{L}\right],\end{aligned} \tag{4.1}$$

where $L=2a$ [18,19].

Then, by using Eq. (2.11), it is immediate to obtain, for $t\to\infty$,

$$\begin{aligned}\mathcal{M}(x,y)=&\left(\frac{2}{L}\right)^2\exp\left(\frac{u_x x}{\sigma_x^2}+\frac{u_y y}{\sigma_y^2}\right)\sum_{m,l=1}^{\infty}\kappa_{m,l}^{-1}\\&\times\sin\left[\frac{m\pi(x+a)}{L}\right]\sin\left[\frac{m\pi a}{L}\right]\sin\left[\frac{l\pi(y+a)}{L}\right]\\&\times\sin\left[\frac{l\pi a}{L}\right],\end{aligned} \tag{4.2}$$

where

$$\kappa_{m,n}=(2L^2)^{-1}[(m^2\sigma_x^2+l^2\sigma_y^2)\pi^2+2L^2(u_x^2/\sigma_x^2+u_y^2/\sigma_y^2)]. \tag{4.3}$$

Equation (4.2) shows that there exists a stationary distribution of the mean number of visits on the lattice S. The existence of a bounded function $\mathcal{M}$ is an important property of this type of random walk and it makes straightforward to compare numerical results obtained by means of Eq. (4.2) with ‘‘experimental’’ values, generated by a Monte Carlo simulation.

First consider an asymmetric random walk without drift but with different variances σ_x^2,σ_y^2. A possible model for such a random walk is given by the transition probabilities

$$\begin{aligned}p(x,y-1|x,y)=p(x,y+1|x,y)=\tfrac{1}{4}+\tfrac{1}{4}\alpha,\\p(x+1,y|x,y)=p(x-1,y|x,y)=\tfrac{1}{4}-\tfrac{1}{4}\alpha,\end{aligned} \tag{4.4}$$

where $\alpha\in[0,1]$ is a parameter determining the asymmetry of the motion. It can immediately be observed that $u_x=u_y=0$ and $\sigma_x^2=(1/2-1/2\alpha)$, $\sigma_y^2=(1/2+1/2\alpha)$, hence by fixing α in Eq. (4.4) one also determines the corresponding values of σ_x^2,σ_y^2 in Eq. (2.11). In the simulation $\mathcal{M}(x,y)$ was the number of visits at site $\mathbf{s}=(x,y)$ averaged over $N=1000$ walkers.

The graph of $\mathcal{M}$ obtained with formula (4.2) is shown in Fig. 1. The agreement between Eq. (4.2) and the Monte Carlo simulation can be seen in detail by considering one-dimensional cuts along the x and y axes, respectively. The results, shown in Fig. 2 for a cut along the y axis demonstrate that indeed Eq. (4.2) provides a good approximation of the process; the same result has been found for cuts along the x axis.

A similar comparison has been carried out for the case of a random walk with drift, i.e., with $u_x,u_y\neq 0$. The Monte Carlo simulation used the transition probabilities

$$\begin{aligned}p(x,y-1|x,y)&=\tfrac{1}{4}-\tfrac{1}{4}\mu,\\p(x-1,y|x,y)&=\tfrac{1}{4}-\tfrac{1}{4}\nu,\\p(x+1,y|x,y)&=\tfrac{1}{4}+\tfrac{1}{4}\nu,\\p(x,y+1|x,y)&=\tfrac{1}{4}+\tfrac{1}{4}\mu,\end{aligned} \tag{4.5}$$

here $u_x=1/2\nu$, $u_y=1/2\mu$, and $\sigma_x^2=\sigma_y^2=1/2$.

As before, the graph of $\mathcal{M}$ is presented in Fig. 3. The comparison with the Monte Carlo simulation is shown in the

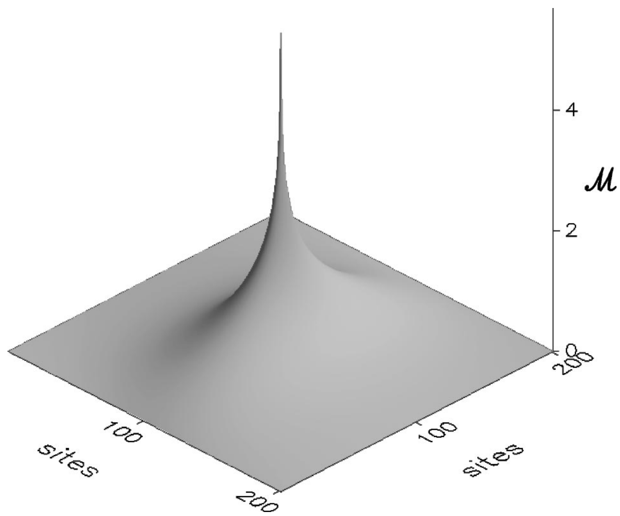

FIG. 1. Graph of $\mathcal{M}$ computed with Eq. (4.2), with $L=201$ sites and m, l ranging from 1 to 1000. Parameter values are $u_x=u_y=0$, $\alpha=0.6$, so that $\sigma_x^2=0.2$, $\sigma_y^2=0.8$. The ripples on the graph are an artifact due to the numerical approximation.

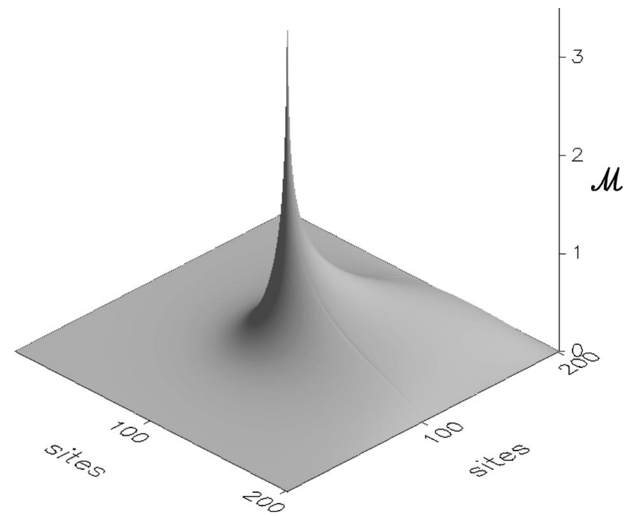

FIG. 3. Graph of $\mathcal{M}$ computed with Eq. (4.2), with $L=201$, and m, l ranging from 1 to 400. Parameter values are $\mu=0.02$, $\nu=0.02$, $u_x=0.01$, $u_y=0.01$. The ripples on the graph are an artifact due to the numerical approximation.

cut along the direction of the drift (see Fig. 4); both figures show that, even for small values of u_x, u_y, the effect of the drift results in a large deviation from the symmetric case.

The analytical formulas derived from the Fokker-Planck equation are in good agreement with the results obtained by the Monte Carlo simulation, but for points close to the origin, as should be expected by the properties of the diffusion approximation discussed earlier.

V. RESTRICTED RANDOM WALK: REFLECTING BOUNDARIES

If S is a square lattice and the boundary is reflecting a simple, or symmetric, the random walk has an uniform invariant distribution [13]: that is,

$$\lim_{k\to\infty} P_k(\mathbf{s}) = \frac{1}{|S|}, \tag{5.1}$$

where $|S|$ is the number of sites of the lattice S. Then, asymptotically, M_n increases linearly with n.

This result can be generalized to the case of asymmetric random walks by solving the Fokker-Plank equation, with the appropriate boundary conditions. Only the one-dimensional case will be considered here, since the results can be easily extended to two dimensions.

The solution of the Fokker-Planck equation is now

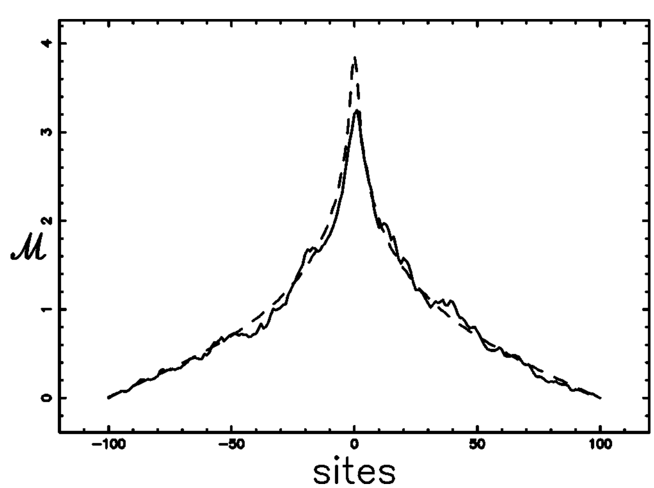

FIG. 2. Cuts along a vertical line (y axis) of the graph of $\mathcal{M}$, obtained with Eq. (4.2) (broken line) and the Monte Carlo simulation (solid line). Here $L=201$ and the range of m, l is [1,100].

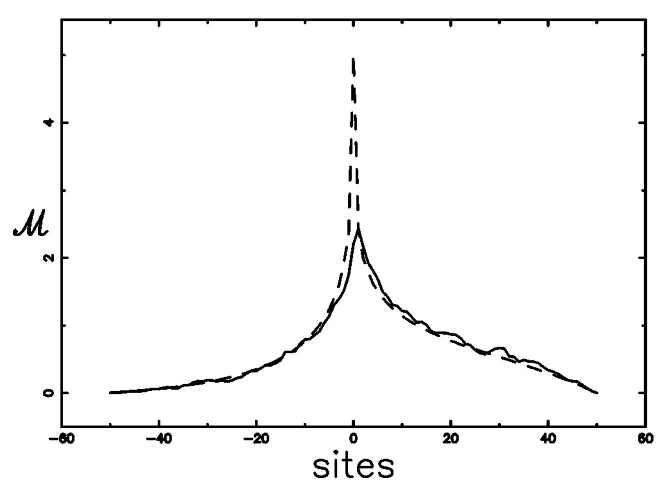

FIG. 4. Cuts along a vertical line (y axis, the direction of the drift) of the graphs of $\mathcal{M}$, as obtained with Eq. (4.2) (broken line) and the Monte Carlo simulation (solid line). Here $u_x=0.01$, $u_y=0$, $L=101$, and the range of m, l is [1,3000].

$$P(x,y;t)=\exp\left[\left(\frac{u_x x}{\sigma_x^2}\right)-\left(\frac{u_x^2 t}{\sigma_x^2}\right)\right]\times\left\{\frac{1}{L}+\left(\frac{2}{L}\right)\sum_{m=1}^{\infty}\exp-(m^2\sigma_x^2\pi^2 t/2L^2)\times\cos\left[\frac{m\pi(x+a)}{L}\right]\cos\left[\frac{m\pi a}{L}\right]\right\}, \tag{5.2}$$

[3,17]; then,

$$M(x,y;t)=\exp\left(\frac{u_x x}{\sigma_x^2}\right)\frac{1}{L}\{t+[1-\exp-(u_x^2 t/\sigma_x^2)]\}+\exp\left(\frac{u_x x}{\sigma_x^2}\right)\sum_{m=1}^{\infty}a_{m,n}^{-1}[1-\exp-(a_{m,n}t)]\times\cos\left[\frac{m\pi(x+a)}{L}\right]\cos\left[\frac{m\pi a}{L}\right], \tag{5.3}$$

with $a_{m,n}=(m^2\sigma_x^2\pi^2+2L^2u_x^2/\sigma_x^2)(2L^2)^{-1}$. Equation (5.3) shows clearly that, for t large, $M(x,\cdot)$ increases linearly in time, the rate of growth depending on x.

The results obtained so far make the case of mixed boundaries easy to deal with. Suppose that one of the sides of S parallel to, say, the y axis is absorbing; $P(x,y;t)$ can be written as $P(x,y;t)=\psi(x,t)\phi(y,t)$, [18], where ψ is given by Eq. (5.2), whereas

$$\phi(x,t)=\frac{2}{L}\exp\left[\left(\frac{u_y y}{\sigma_y^2}\right)-\left(\frac{u_y^2 t}{\sigma_y^2}\right)\right]\times\sum_{l=0}^{\infty}\exp-[(2l+1)(l^2\sigma_y^2\pi^2 t/2L^2)]C_{l+1}(y), \tag{5.4}$$

C_{l+1} being a product of cosines [3]. It is then apparent that $P(x,y;t)t\to 0$ as $t\to\infty$ and $\mathcal{M}$ is, again, a bounded function of x and y that can be easily computed.

VI. CONCLUSION

The results of this note clarify how the mean number of times a site $\mathbf{s}=(x,y)$ is visited by a random walker moving on a square lattice depends on the type of walk considered and on the boundary conditions.

In the case of an unrestricted random walk M_n scales logarithmically with n. If the random walk is restricted by a reflecting boundary $M(x,y;\cdot)$ grows at a faster rate, as should be expected since the walker moves on a finite domain and thus the probability of a given site to be visited is greater.

The case of absorbing boundaries is more interesting in that there exists a stationary state, characterized by $\mathcal{M}$. Indeed, M_n, or M, can be thought of as a memory left by the random walk on the lattice. If no part of the boundary is absorbing, all sites will be visited infinitely often and that can be interpreted as a loss of memory, in that all information about the characteristics of the random walk is lost as time increases, whereas this information is retained by $\mathcal{M}$ if at least one side of the boundary is absorbing.

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